

LINEAR DIFFERENCE EQUATIONS WITH A TRANSITION POINT AT THE ORIGIN

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ABSTRACT. A pair of linearly independent asymptotic solutions are constructed for the second-order linear difference equation

$$P_{n+1}(x) - (A_n x + B_n)P_n(x) + P_{n-1}(x) = 0,$$

where A_n and B_n have asymptotic expansions of the form

$$A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s},$$

with $\theta \neq 0$ and $\alpha_0 \neq 0$ being real numbers, and $\beta_0 = \pm 2$. Our result hold uniformly for the scaled variable t in an infinite interval containing the transition point $t_1 = 0$, where $t = (n + \tau_0)^{-\theta} x$ and τ_0 is a small shift. In particular, it is shown how the Bessel functions J_ν and Y_ν get involved in the uniform asymptotic expansions of the solutions to the above three-term recurrence relation. As an illustration of the main result, we derive a uniform asymptotic expansion for the orthogonal polynomials associated with the Laguerre-type weight $x^\alpha \exp(-q_m x^m)$, $x > 0$, where m is a positive integer, $\alpha > -1$ and $q_m > 0$.

Keywords: difference equation; transition point; three-term recurrence relations; uniform asymptotic expansions; Bessel functions; orthogonal polynomials

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1. INTRODUCTION

Orthogonal polynomials play an important role in many branches of mathematical physics, for instance, quantum mechanics, scattering theory and statistical mechanics. A major topic in orthogonal polynomials is the study of their asymptotic behavior as the degree grows to infinity. Since the classical orthogonal polynomials (Hermite, Laguarre and Jacobi) all satisfy a second-order linear differential equation, their asymptotic behavior can be obtained from the WKB approximation or the turning point theory [15]. For discrete orthogonal polynomials (e.g., Charlier, Meixner and Krawtchouk), one can use their generating function to obtain a Cauchy integral representation and then apply the steepest descent method or its extensions [15, 23]. However, there are orthogonal polynomials that neither satisfy any differential equation nor have integral representations. A powerful method, known as the nonlinear steepest

descent method for Riemann-Hilbert problems, has recently been developed that can be applied to such polynomials. Papers that deserve special mention include Deift *et al.* [9], Bleher and Its [6], Kriecherbauer and McLaughlin [14], Baik *et al.* [2], Ou and Wong [17] and Zhou *et al.* [27]. Despite the great success achieved by this powerful method, the Riemann-Hilbert analysis depends heavily on the analyticity of the weight functions, and the argument and results obtained by the method often appear in a very complicated manner.

In our view, a more natural approach to derive asymptotic expansions for orthogonal polynomials is to develop an asymptotic theory for linear second-order difference equations, just in the same way as Langer, Cherry, Olver had done for linear second-order differential equations; see the definitive book by Olver [15]. Our view is based on the fact that any sequence of orthogonal polynomials satisfies a three-term recurrence relation of the form

$$(1.1) \quad p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x), \quad n = 1, 2, \dots,$$

where a_n , b_n and c_n are constants; see [18, p.43]. If x is a fixed number, then this recurrence relation is equivalent to a second-order linear difference equation of the form

$$(1.2) \quad y(n+2) + n^p a(n)y(n+1) + n^q b(n)y(n) = 0,$$

where p and q are integers. We assume that the coefficient functions $a(n)$ and $b(n)$ have asymptotic expansions

$$(1.3) \quad a(n) \sim \sum_{s=0}^{\infty} \frac{a_s}{n^s}, \quad b(n) \sim \sum_{s=0}^{\infty} \frac{b_s}{n^s},$$

where $a_0 \neq 0$ and $b_0 \neq 0$. When $p = q = 0$, asymptotic solutions to this equation are classified by the roots of the *characteristic equation*

$$(1.4) \quad \rho^2 + a_0 \rho + b_0 = 0.$$

If $\rho_1 \neq \rho_2$, *i.e.*, $a_0^2 \neq 4b_0$, then Birkhoff [3] showed that (1.2) has two linearly independent solutions, both of the form

$$(1.5) \quad y(n) \sim \rho^n n^\alpha \sum_{s=0}^{\infty} \frac{c_s}{n^s}, \quad n \rightarrow \infty,$$

where

$$(1.6) \quad \alpha = -\frac{a_1 \rho + b_1}{2\rho^2 + a_0 \rho} = \frac{a_1 \rho + b_1}{2b_0 + a_0 \rho},$$

and the coefficients c_s can be determined recursively. This construction fails when and only when $\rho_1 = \rho_2$, *i.e.*, when $a_0^2 = 4b_0$. Motivated by the terminologies in differential equations [15, p.230], we shall call series of the form (1.5) *normal series*. If $\rho_1 = \rho_2$, but $2b_1 \neq a_0 a_1$, then Adams [1] constructed two linearly independent power series solutions, which we will call *subnormal series*.

Adams also studied the case when $2b_1 = a_0a_1$, but his analysis was incomplete, as pointed out by Birkhoff [4]. A powerful asymptotic theory for difference equations was later given by Birkhoff [4], Birkhoff and Trjitzinsky [5]. However, their analysis has been considered too complicated and even impenetrable. A more accessible approach to the results mentioned above has been given in more recent years by Wong and Li [24, 25]. In their first paper [24], they presented recursive formulas for the coefficients in the power series solutions when $p = q = 0$ in (1.2), obtained both norm and subnormal series solutions. Their second paper [25] dealt with the more general case, namely, equation (1.2) with the exponents p and q being not both zero. Numerically computable error bounds for these asymptotic series can be found in our recent paper [7].

Returning to the three-term recurrence relation (1.1), we note that the results mentioned thus far apply only when x is a fixed number. When x is a variable and allowed to vary, the roots of the characteristic equation associated with (1.1) may coalesce, and normal series solutions may collapse to become subnormal series. Not much work has been done in this area until just recently. In a series of papers [20, 21, 22], Wang and Wong have derived asymptotic expansions for the solutions to (1.1), which hold “uniformly” for x in infinite intervals. They first define a sequence $\{K_n\}$ recursively by $K_{n+1}/K_n = c_n$, with K_0 and K_1 depending on the particular sequence of polynomials, where c_n is one of the coefficients in equation (1.1). Then they put $A_n \equiv a_n K_n/K_{n+1}$, $B_n \equiv b_n K_n/K_{n+1}$ and $P_n(x) \equiv p_n(x)/K_n$, so that (1.1) becomes

$$(1.7) \quad P_{n+1}(x) - (A_n x + B_n)P_n(x) + P_{n-1}(x) = 0.$$

The coefficients A_n and B_n are assumed to have asymptotic expansions of the form

$$(1.8) \quad A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s},$$

where θ is a real number and $\alpha_0 \neq 0$. If τ_0 is a constant and $N := n + \tau_0$, then the expansions in (1.8) can of course be recasted in the form

$$(1.9) \quad A_n \sim N^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha'_s}{N^s}, \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{N^s}.$$

In (1.7), we now set $x := N^\theta t$ and $P_n = \rho^n$. Substituting (1.9) into (1.7) and letting $n \rightarrow \infty$ (and hence $N \rightarrow \infty$), we are led to the *characteristic equation*

$$(1.10) \quad \rho^2 - (\alpha'_0 t + \beta'_0) \rho + 1 = 0.$$

(Note that $\alpha'_0 = \alpha_0$ and $\beta'_0 = \beta_0$). The roots of this equation coincide when $t = t_i$, $i = 1, 2$, where

$$(1.11) \quad \alpha'_0 t_1 + \beta'_0 = +2, \quad \alpha'_0 t_2 + \beta'_0 = -2.$$

The values of t_1 and t_2 play an important role in the asymptotic theory of the three-term recurrence relation (1.7), and they correspond to the transition points (*i.e.*, turning points or singularities) occurring in differential equations [15]. For this reason, Wang and Wong [21] also called them *transition points* of difference equations. Furthermore, they indicated that in terms of the exponent θ in (1.8) and the transition point t_1 , there are three cases to be considered; namely, (i) $\theta \neq 0$ and $t_1 \neq 0$, (ii) $\theta \neq 0$ and $t_1 = 0$, and (iii) $\theta = 0$. In [21], they considered case (i), which turns out to correspond to the turning-point problem for second-order linear differential equations [15, p.392]. They also showed that the asymptotic expansions of the solutions involve the Airy functions $\text{Ai}(\cdot)$, $\text{Bi}(\cdot)$ and their derivatives, and furthermore that the expansions hold uniformly for t in the infinite interval $[0, \infty)$. In [22], they studied case (iii), and found that the approximants of the asymptotic solutions are Bessel functions or modified Bessel functions. We should mention that, the WKB approximations for difference equations have also been studied in recent years by Costin and Costin [8], Geronimo, Bruno and Van Assche [13], Geronimo [12], etc. All of these authors dealt with only case (i), *i.e.*, the Airy-type expansions.

The purpose of this paper is to provide a solution to the problem in case (ii), which is the only case that have been left out in Wang and Wong's investigation of the asymptotic solutions to the three-term recurrence relation (1.7). In this case, we shall show that the approximants of the asymptotic solutions are also Bessel functions or modified Bessel functions, but the order of these Bessel functions and the transformation used (*i.e.*, $\zeta(t)$ in (4.9) below) differ from the one used in case (iii). As an illustration of the main result, we derive a uniform asymptotic expansion for the orthogonal polynomials associated with the Laguerre-type weight $x^\alpha \exp(-q_m x^m)$, $x > 0$, where m is a positive integer, $\alpha > -1$ and $q_m > 0$.

2. MOTIVATION LEADING TO THE EXPANSION

Turing point (or transition point) theory for second-order linear difference equations was well established in [21, 22]. We simply state the procedure below.

Let $\tau_0 := -\alpha_1/(\alpha_0\theta)$ and $N := n + \tau_0$. Then,

$$(2.1) \quad A_n x + B_n = n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s} x + \sum_{s=0}^{\infty} \frac{\beta_s}{n^s} := \sum_{s=0}^{\infty} \frac{\alpha'_s t + \beta'_s}{N^s}.$$

A slight computation yields

$$(2.2) \quad \alpha'_0 = \alpha_0, \quad \alpha'_1 = 0, \quad \beta'_0 = \beta_0, \quad \beta'_1 = \beta_1, \quad \beta'_2 = \beta_2 + \beta_1 \tau_0.$$

If $\beta_0 = 2$ (or -2), it follows from (1.11) that one of the transition points is zero. Without loss of generality, we may assume that $t_1 = 0 < t_2$ and $\beta_0 = 2$,

$\alpha_0 < 0$. (For other cases, see Remark 1 below.) Through out this paper, we assume that $\beta_1 = 0$, so that $\beta'_1 = 0$ and

$$(2.3) \quad \alpha'_1 t_1 + \beta'_1 = \beta_1 = 0.$$

This assumption was used in the previous papers on transition point theory; see [21, (2.7)] and [22, (2.5)]. (In most of the classical cases, $\beta_1 = 0$ and hence $\beta'_1 = 0$. Also, we note that in [10] Dingle and Morgan have assumed a more strict condition that $\alpha_{2s+1} = \beta_{2s+1} = 0$ for $s = 0, 1, 2, \dots$.)

In what follows we will investigate the uniform asymptotic expansions of solutions to (1.7) near the transition point $t_1 = 0$. To this end, we seek a formal solution to (1.7) of the form

$$(2.4) \quad P_n(x) = \sum_{s=0}^{\infty} \chi_s(\xi) N^{-s}, \quad \xi = N^2 \eta(t),$$

for t in a neighbourhood of $t_1 = 0$, where $\eta(t)$ is an increasing function with $\eta(0) = 0$. Since $x := N^\theta t$ is fixed, when we change n to $n \pm 1$, we must in the same time change t to t_\pm , where

$$(2.5) \quad t_\pm = \left(1 \pm \frac{1}{N}\right)^{-\theta} t.$$

As a consequence, it follows from (2.4) that

$$(2.6) \quad P_{n\pm 1}(x) = \sum_{s=0}^{\infty} \chi_s \left[(N \pm 1)^2 \eta \left(\left(1 \pm \frac{1}{N}\right)^{-\theta} t \right) \right] (N \pm 1)^{-s}.$$

For convenience, we also introduce the notations

$$(2.7) \quad Q_\pm(\xi) := (N \pm 1)^2 \eta \left[\left(1 \pm \frac{1}{N}\right)^{-\theta} t \right]$$

and

$$(2.8) \quad \Psi(\xi) := A_n x + B_n = \sum_{s=0}^{\infty} \frac{\alpha'_s t + \beta'_s}{N^s} := \sum_{s=0}^{\infty} \frac{T_s(\xi)}{N^s}.$$

Substituting (2.4) and (2.6) into (1.7), we get

$$(2.9) \quad \sum_{s=0}^{\infty} \left\{ \chi_s[Q_+(\xi)] \left(1 + \frac{1}{N}\right)^{-s} + \chi_s[Q_-(\xi)] \left(1 - \frac{1}{N}\right)^{-s} - \Psi(\xi) \chi_s(\xi) \right\} \frac{1}{N^s} = 0.$$

Noting that $t = \eta^{-1}(N^{-2}\xi)$, Taylor expansion gives

$$(2.10) \quad t = \sum_{j=0}^{\infty} \frac{\eta^{-1(j)}(0)}{j!} \left(\frac{\xi}{N^2} \right)^j = \frac{1}{\eta'(0)} \frac{\xi}{N^2} + \mathcal{O}(N^{-4}).$$

Substituting the last equation into (2.8) yields,

$$(2.11) \quad T_0(\xi) = \beta'_0 = \beta_0, \quad T_1(\xi) = \beta'_1 = 0, \quad T_2(\xi) = \frac{\alpha'_0 \xi}{\eta'(0)} + \beta'_2.$$

By expanding $\eta(t_{\pm})$ at t , we obtain from (2.5) and (2.7)

$$(2.12) \quad Q_{\pm}(\xi) = \xi \mp \frac{(\theta - 2)\xi}{N} + \frac{(\theta^2 - 3\theta + 2)\xi}{2N^2} + \mathcal{O}(N^{-3}) := \sum_{s=0}^{\infty} \frac{Q_s^{\pm}(\xi)}{N^s}.$$

In equation (2.9), the coefficient of N^{-s} vanishes for each $s \geq 0$; *i.e.*,

$$(2.13) \quad \chi_s[Q_+(\xi)] \left(1 + \frac{1}{N}\right)^{-s} + \chi_s[Q_-(\xi)] \left(1 - \frac{1}{N}\right)^{-s} - \Psi(\xi) \chi_s(\xi) = 0.$$

In particular, for $s = 0$, expanding the first two terms at ξ and using (2.8) and (2.12) show that $\chi_0(\xi)$ satisfies Bessel's equation

$$(2.14) \quad \frac{d^2 \chi_0}{d\xi^2} + \left(\frac{\theta - 1}{\theta - 2}\right) \frac{1}{\xi} \frac{d\chi_0}{d\xi} - \frac{1}{(\theta - 2)^2 \xi^2} \left(\frac{\alpha'_0 \xi}{\eta'(0)} + \beta'_2\right) \chi_0 = 0,$$

where $\theta \neq 2$. For simplicity, we shall assume $0 < \theta < 2$. The analysis for the case $\theta < 0$ or $\theta > 2$ is very similar. Thus, $\chi_0(\xi)$ can be expressed in terms of Bessel functions:

$$\chi_0(\xi) = C_1 \xi^{\frac{1}{2(2-\theta)}} J_{\nu}(b\xi^{1/2}) + C_2 \xi^{\frac{1}{2(2-\theta)}} Y_{\nu}(b\xi^{1/2}),$$

where C_1, C_2 are two constants, $b = \sqrt{\frac{-4\alpha'_0}{(\theta-2)^2\eta'(0)}}$ and

$$(2.15) \quad \nu = \sqrt{\frac{1 + 4\beta'_2}{(\theta - 2)^2}}.$$

Moreover, each of the subsequent coefficient functions $\chi_s(\xi)$, $s = 1, 2, \dots$, in (2.4) satisfies an inhomogeneous Bessel equation. This suggests that instead of (2.4), we might try the formal series solution

$$(2.16) \quad P_n(x) = Z_{\nu}(N\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + Z_{\nu+1}(N\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}$$

motivated from the differential equation theory, where we have set $\zeta(t) = [\eta(t)]^{\frac{1}{2}}$ to simplify the notations. Here $\zeta(t) > 0$ for $t > 0$ and $i\zeta(t) < 0$ for $t < 0$. In (2.16), $Z_{\nu}(\cdot)$ could be any solution of Bessel's equation

$$(2.17) \quad y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

We state the main result of this paper in the following theorem.

Theorem 1. Assume that the coefficients A_n and B_n in the recurrence relation (1.7) are real, and have asymptotic expansions given in (1.8) with $\theta \neq 0, 2$ and $\beta_0 = 2$. Let $t_1 = 0$ be a transition point defined in (1.11), and the function $\zeta(t)$ be given as in (4.9) and (4.10). Then, for each nonnegative integer p , (1.7) has a pair of linearly independent solutions,

$$(2.18) \quad P_n(N^\theta t) = N^{\frac{1}{2}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} \left[J_\nu(N\zeta) \sum_{s=0}^p \frac{\tilde{A}_s(\zeta)}{N^s} + J_{\nu+1}(N\zeta) \sum_{s=0}^p \frac{\tilde{B}_s(\zeta)}{N^s} + \varepsilon_n^p \right]$$

and

$$(2.19) \quad Q_n(N^\theta t) = N^{\frac{1}{2}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} \left[W_\nu(N\zeta) \sum_{s=0}^p \frac{\tilde{A}_s(\zeta)}{N^s} + W_{\nu+1}(N\zeta) \sum_{s=0}^p \frac{\tilde{B}_s(\zeta)}{N^s} + \delta_n^p \right],$$

where $W_\nu(x) := Y_\nu(x) - iJ_\nu(x)$, $N = n + \tau_0$, $\tau_0 = -\alpha_1/(\alpha_0\theta)$ and ν is given in (2.15). The error terms satisfy

$$(2.20) \quad |\varepsilon_n^p| \leq \frac{M_p}{N^{p+1}} [|J_\nu(N\zeta)| + |J_{\nu+1}(N\zeta)|]$$

and

$$(2.21) \quad |\delta_n^p| \leq \frac{M_p}{N^{p+1}} [|W_\nu(N\zeta)| + |W_{\nu+1}(N\zeta)|]$$

for $-\infty < t \leq t_2 - \sigma$, M_p being a positive constant and σ being an arbitrary positive constant. The coefficients $\tilde{A}_0(\zeta) = 1$ and $\tilde{B}_0(\zeta) = 0$ and other $\tilde{A}_s(\zeta)$ and $\tilde{B}_s(\zeta)$ can be determined successively from their predecessors; see (4.26) and (4.27).

Remark 1. For the case $\alpha_0 > 0$ and $\beta_0 = -2$, the two transition points t_1 and t_2 again satisfy $t_1 = 0 < t_2$. Put $\mathcal{P}_n(x) := (-1)^n P_n(x)$. Theorem 1 then applies to $\mathcal{P}_n(x)$. If the two transition points satisfy $t_2 < 0 = t_1$, we set $x := -N^\theta t$, instead of $x := N^\theta t$. Theorem 1 again holds with $P_n(N^\theta t)$ replaced by $P_n(-N^\theta t)$ (in the case $\alpha_0 > 0$, $\beta_0 = 2$) or $(-1)^n P_n(-N^\theta t)$ (in the case $\alpha_0 < 0$, $\beta_0 = -2$).

3. A PRELIMINARY LEMMA

As we have noted in above sections, $x := N^\theta t$ is fixed in equation (1.7). Hence, as N being replaced by $N \pm 1$, the two functions $Z_\nu(N\zeta)$ and $Z_{\nu+1}(N\zeta)$ in (2.16) automatically change to $Z_\nu[(N \pm 1)\zeta(t_\pm)]$ and $Z_{\nu+1}[(N \pm 1)\zeta(t_\pm)]$,

where t_{\pm} are given in (2.5). The following lemma plays a crucial role in the derivation of the formal series solution in (2.16).

Lemma 3.1. *Let $Z_{\nu}(x)$ be any solution of Bessel's equation*

$$(3.1) \quad y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) = 0.$$

Then, we have

$$(3.2) \quad Z_{\nu} \{(N \pm 1)\zeta(t_{\pm})\} = Z_{\nu}(N\zeta)G_{\pm}\left(\zeta, \frac{1}{N}\right) + Z_{\nu+1}(N\zeta)H_{\pm}\left(\zeta, \frac{1}{N}\right)$$

and

$$(3.3) \quad Z_{\nu+1} \{(N \pm 1)\zeta(t_{\pm})\} = Z_{\nu}(N\zeta)L_{\pm}\left(\zeta, \frac{1}{N}\right) + Z_{\nu+1}(N\zeta)K_{\pm}\left(\zeta, \frac{1}{N}\right),$$

where

$$(3.4) \quad G_{\pm}\left(\zeta, \frac{1}{N}\right) \sim \sum_{s=0}^{\infty} (\pm 1)^s \frac{G_s(\zeta)}{N^s}, \quad H_{\pm}\left(\zeta, \frac{1}{N}\right) \sim \sum_{s=0}^{\infty} (\pm 1)^{s-1} \frac{H_s(\zeta)}{N^s}$$

and

$$(3.5) \quad L_{\pm}\left(\zeta, \frac{1}{N}\right) \sim \sum_{s=0}^{\infty} (\pm 1)^{s-1} \frac{L_s(\zeta)}{N^s}, \quad K_{\pm}\left(\zeta, \frac{1}{N}\right) \sim \sum_{s=0}^{\infty} (\pm 1)^s \frac{K_s(\zeta)}{N^s},$$

the expansions being uniformly valid with respect to bounded t .

Proof. Let

$$(3.6) \quad w(N, \zeta, u) = \left(1 + \frac{u}{N}\right)^{\frac{1}{2}} Z_{\nu}[(N + u)\zeta].$$

A straightforward calculation gives

$$(3.7) \quad \frac{\partial w}{\partial u} = \frac{\frac{1}{2} + \nu}{N} \left(1 + \frac{u}{N}\right)^{-\frac{1}{2}} Z_{\nu} - \left(1 + \frac{u}{N}\right)^{\frac{1}{2}} \zeta Z_{\nu+1}$$

and

$$(3.8) \quad \frac{\partial^2 w}{\partial u^2} = \left(\frac{\nu^2 - \frac{1}{4}}{(N + u)^2} - \zeta^2\right) w,$$

where we have made use of the difference-differential relations of Bessel functions

$$Z'_{\nu}(x) = \frac{\nu}{x} Z_{\nu}(x) - Z_{\nu+1}(x)$$

and

$$Z'_{\nu+1}(x) = Z_{\nu}(x) - \frac{\nu + 1}{x} Z_{\nu+1}(x);$$

see [16, §10.6]. The Taylor expansion of $w(N, \zeta, u)$ at $u = 0$ is

$$(3.9) \quad w(N, \zeta, u) = w(N, \zeta, 0) + u \frac{\partial w}{\partial u}(N, \zeta, 0) + \frac{u^2}{2!} \frac{\partial^2 w}{\partial u^2}(N, \zeta, 0) + \cdots$$

In view of (3.6)-(3.8), the above series can be rearranged as

$$(3.10) \quad \left(1 + \frac{u}{N}\right)^{\frac{1}{2}} Z_\nu [(N+u)\zeta] = Z_\nu(N\zeta)\tilde{G}(N; \zeta, u) + Z_{\nu+1}(N\zeta)\tilde{H}(N; \zeta, u),$$

where

$$(3.11) \quad \tilde{G}(N; \zeta, 0) = 1, \quad \tilde{H}(N; \zeta, 0) = 0,$$

$$(3.12) \quad \frac{\partial \tilde{G}}{\partial u}(N; \zeta, 0) = \frac{\nu + \frac{1}{2}}{N} \quad \text{and} \quad \frac{\partial \tilde{H}}{\partial u}(N; \zeta, 0) = -\zeta.$$

Differentiating (3.10) with respect to u twice yields

$$(3.13) \quad \frac{\partial^2 \tilde{G}}{\partial u^2} = \left(\frac{\nu^2 - \frac{1}{4}}{(N+u)^2} - \zeta^2 \right) \tilde{G}, \quad \tilde{G}|_{u=0} = 1, \quad \frac{\partial \tilde{G}}{\partial u} \Big|_{u=0} = \frac{\frac{1}{2} + \nu}{N}$$

and

$$(3.14) \quad \frac{\partial^2 \tilde{H}}{\partial u^2} = \left(\frac{\nu^2 - \frac{1}{4}}{(N+u)^2} - \zeta^2 \right) \tilde{H}, \quad \tilde{H}|_{u=0} = 0, \quad \frac{\partial \tilde{H}}{\partial u} \Big|_{u=0} = -\zeta.$$

We try formal solutions to (3.13) and (3.14) of the following forms

$$(3.15) \quad \tilde{G}(N; \zeta, u) \sim \sum_{s=0}^{\infty} \frac{\tilde{G}_s(\zeta, u)}{N^s} \quad \text{and} \quad \tilde{H}(N; \zeta, u) \sim \sum_{s=0}^{\infty} \frac{\tilde{H}_s(\zeta, u)}{N^s},$$

where the coefficients \tilde{G}_s can be determined recursively by the equations

$$(3.16) \quad \frac{\partial^2 \tilde{G}_0}{\partial u^2} = -\zeta^2 \tilde{G}_0, \quad \tilde{G}_0(0, \zeta) = 1, \quad \frac{\partial \tilde{G}_0}{\partial u}(0, \zeta) = 0,$$

$$(3.17) \quad \frac{\partial^2 \tilde{G}_1}{\partial u^2} = -\zeta^2 \tilde{G}_1, \quad \tilde{G}_1(0, \zeta) = 0, \quad \frac{\partial \tilde{G}_1}{\partial u}(0, \zeta) = \frac{1}{2} + \nu,$$

$$(3.18) \quad \frac{\partial^2 \tilde{G}_s}{\partial u^2} = -\zeta^2 \tilde{G}_s + \left(\nu^2 - \frac{1}{4} \right) \sum_{j=2}^s (j-1)(-u)^{j-2} \tilde{G}_{s-j}, \quad s \geq 2,$$

and

$$(3.19) \quad \tilde{G}_s(0, \zeta) = 0, \quad \frac{\partial \tilde{G}_s}{\partial u}(0, \zeta) = 0, \quad s \geq 2.$$

The solutions of (3.16)-(3.19) are

$$(3.20) \quad \tilde{G}_0 = \cos(\zeta u), \quad \tilde{G}_1 = \frac{\frac{1}{2} + \nu}{\zeta} \sin(\zeta u)$$

and

$$(3.21) \quad \tilde{G}_s = \frac{\nu^2 - \frac{1}{4}}{\zeta} \int_0^u \left(\sum_{j=2}^s (-1)^j (j-1) \phi^{j-2} \tilde{G}_{s-j} \right) \sin((u-\phi)\zeta) d\phi$$

for $s \geq 2$. Similarly, we also have

$$(3.22) \quad \tilde{H}_0 = -\sin(\zeta u), \quad \tilde{H}_1 = 0$$

and

$$(3.23) \quad \tilde{H}_s = \frac{\nu^2 - \frac{1}{4}}{\zeta} \int_0^u \left(\sum_{j=2}^s (-1)^j (j-1) \phi^{j-2} \tilde{H}_{s-j} \right) \sin((u-\phi)\zeta) d\phi$$

for $s \geq 2$. By induction, we can prove that when $i\zeta < 0$,

$$|\tilde{G}_s(u, \zeta)| \leq (|\nu| + 1)^s |u|^s \tilde{G}_0(u, \zeta), \quad |\tilde{H}_s(u, \zeta)| \leq (|\nu| + 1)^s |u|^s |\tilde{H}_0(u, \zeta)|,$$

and that when $\zeta > 0$,

$$|\tilde{G}_s(u, \zeta)| \leq (|\nu| + 1)^s |u|^s, \quad |\tilde{H}_s(u, \zeta)| \leq (|\nu| + 1)^s |u|^s.$$

Thus, the formal series in (3.15) are uniformly convergent for any bounded u and sufficiently large N .

To show that (3.2) follows from (3.10), we simply set

$$(N + u)\zeta = (N \pm 1)\zeta(t_{\pm})$$

i.e., we choose

$$(3.24) \quad u = u_{\pm} := (N \pm 1) \frac{\zeta(t_{\pm})}{\zeta(t)} - N.$$

Furthermore, we let

$$(3.25) \quad G_{\pm}\left(\zeta, \frac{1}{N}\right) = \left(1 + \frac{u_{\pm}}{N}\right)^{-\frac{1}{2}} \tilde{G}(N; \zeta, u_{\pm})$$

and

$$(3.26) \quad H_{\pm}\left(\zeta, \frac{1}{N}\right) = \left(1 + \frac{u_{\pm}}{N}\right)^{-\frac{1}{2}} \tilde{H}(N; \zeta, u_{\pm}).$$

Recall that t_{\pm} is given in (2.5). Expand t_{\pm} into power series in $\frac{1}{N}$, and then expand $\zeta(t_{\pm})$ into Taylor series at t and rearrange them also into power series in $\frac{1}{N}$. From (3.24), it follows that

$$(3.27) \quad u_{\pm} = \sum_{s=0}^{\infty} (\pm 1)^{s+1} \frac{u_s}{N^s},$$

where

$$(3.28) \quad u_0 = 1 - \theta t \frac{\zeta'(t)}{\zeta(t)}, \quad u_1 = \frac{\theta(\theta-1)t}{2} \frac{\zeta'(t)}{\zeta(t)} + \frac{\theta^2 t^2}{2} \frac{\zeta''(t)}{\zeta(t)}.$$

Substituting (3.27) into (3.15), we obtain

$$(3.29) \quad G_{\pm}\left(\zeta, \frac{1}{N}\right) = \left(1 + \frac{u_{\pm}}{N}\right)^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{\tilde{G}_s(\zeta, u_{\pm})}{N^s} := \sum_{s=0}^{\infty} \frac{G_s^{\pm}(\zeta)}{N^s},$$

$$(3.30) \quad H_{\pm}\left(\zeta, \frac{1}{N}\right) = \left(1 + \frac{u_{\pm}}{N}\right)^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{\tilde{H}_s(\zeta, u_{\pm})}{N^s} := \sum_{s=0}^{\infty} \frac{H_s^{\pm}(\zeta)}{N^s}.$$

Simple calculation shows that the first two terms in the expansions of (3.29) and (3.30) are, respectively, given by

$$(3.31) \quad G_0^{\pm}(\zeta) = \cos(\zeta u_0), \quad G_1^{\pm}(\zeta) = \pm \frac{\frac{1}{2} + \nu - \zeta^2 u_1}{\zeta} \sin(\zeta u_0) \mp \frac{1}{2} u_0 \cos(\zeta u_0)$$

and

$$(3.32) \quad H_0^{\pm}(\zeta) = \mp \sin(\zeta u_0), \quad H_1^{\pm}(\zeta) = -u_1 \zeta \cos(\zeta u_0) + \frac{u_0}{2} \sin(\zeta u_0).$$

Equation (3.3) and the expansions in (3.5) can be established in similar manners. Moreover, we have

$$(3.33) \quad K_0^{\pm}(\zeta) = \cos(\zeta u_0), \quad K_1^{\pm}(\zeta) = \mp \frac{\frac{1}{2} + \nu + \zeta^2 u_1}{\zeta} \sin(\zeta u_0) \mp \frac{u_0}{2} \cos(\zeta u_0)$$

and

$$(3.34) \quad L_0^{\pm}(\zeta) = \pm \sin(\zeta u_0), \quad L_1^{\pm}(\zeta) = u_1 \zeta \cos(\zeta u_0) - \frac{u_0}{2} \sin(\zeta u_0).$$

By mathematical induction, one can show that

$$(3.35) \quad G_s^{-}(\zeta) = (-1)^s G_s^{+}(\zeta), \quad H_s^{-}(\zeta) = (-1)^{s-1} H_s^{+}(\zeta)$$

and

$$(3.36) \quad K_s^{-}(\zeta) = (-1)^s K_s^{+}(\zeta), \quad L_s^{-}(\zeta) = (-1)^{s-1} L_s^{+}(\zeta).$$

When we consider the case ‘+’, for simplicity we shall drop the subscripts and superscripts in all above equations. Thus, (3.35) and (3.36) give

$$(3.37) \quad G_{\pm}\left(\zeta, \frac{1}{N}\right) = G\left(\zeta, \pm \frac{1}{N}\right), \quad H_{\pm}\left(\zeta, \frac{1}{N}\right) = \pm H\left(\zeta, \pm \frac{1}{N}\right)$$

and

$$(3.38) \quad K_{\pm}\left(\zeta, \frac{1}{N}\right) = K\left(\zeta, \pm \frac{1}{N}\right), \quad L_{\pm}\left(\zeta, \frac{1}{N}\right) = \pm L\left(\zeta, \pm \frac{1}{N}\right).$$

This completes the proof of Lemma 3.1. \square

4. FORMAL ASYMPTOTIC SOLUTIONS

Let $\zeta^2(t) = \eta(t)$ be an increasing function with $\zeta(0) = 0$; see the sentence following (2.16). We try a formal series solution to (1.7) of the form

$$(4.1) \quad P_n(N^{\theta}t) = Z_{\nu}(N\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + Z_{\nu+1}(N\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s},$$

where Z_ν is a solution of Bessel's equation. For convenience, we put

$$(4.2) \quad A\left(\zeta, \frac{1}{N}\right) := \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s}, \quad B\left(\zeta, \frac{1}{N}\right) := \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}$$

and

$$(4.3) \quad \Psi\left(t, \frac{1}{N}\right) := A_n x + B_n = \sum_{s=0}^{\infty} \frac{\alpha'_s t + \beta'_s}{N^s};$$

cf. (2.8). Using (2.16) and Lemma 3.1, we have

$$(4.4) \quad \begin{aligned} P_{n\pm 1}(x) = & Z_\nu(N\zeta) \left\{ G\left(\zeta, \pm \frac{1}{N}\right) A\left(\zeta(t_\pm), \frac{1}{N \pm 1}\right) \right. \\ & \left. \pm L\left(\zeta, \pm \frac{1}{N}\right) B\left(\zeta(t_\pm), \frac{1}{N \pm 1}\right) \right\} \\ & + Z_{\nu+1}(N\zeta) \left\{ K\left(\zeta, \pm \frac{1}{N}\right) B\left(\zeta(t_\pm), \frac{1}{N \pm 1}\right) \right. \\ & \left. \pm H\left(\zeta, \pm \frac{1}{N}\right) A\left(\zeta(t_\pm), \frac{1}{N \pm 1}\right) \right\}; \end{aligned}$$

see also (3.37) and (3.38). Substituting (4.1)–(4.4) into (1.7) and matching the coefficients of Z_ν and $Z_{\nu+1}$, we get

$$(4.5) \quad \begin{aligned} & G\left(\zeta, \frac{1}{N}\right) A\left[\zeta(t_+), \frac{1}{N+1}\right] + L\left(\zeta, \frac{1}{N}\right) B\left[\zeta(t_+), \frac{1}{N+1}\right] \\ & + G\left(\zeta, \frac{-1}{N}\right) A\left[\zeta(t_-), \frac{1}{N-1}\right] - L\left(\zeta, \frac{-1}{N}\right) B\left[\zeta(t_-), \frac{1}{N-1}\right] \\ & - \Psi\left(t, \frac{1}{N}\right) A\left(\zeta, \frac{1}{N}\right) = 0 \end{aligned}$$

and

$$(4.6) \quad \begin{aligned} & H\left(\zeta, \frac{1}{N}\right) A\left[\zeta(t_+), \frac{1}{N+1}\right] + K\left(\zeta, \frac{1}{N}\right) B\left[\zeta(t_+), \frac{1}{N+1}\right] \\ & - H\left(\zeta, \frac{-1}{N}\right) A\left[\zeta(t_-), \frac{1}{N-1}\right] + K\left(\zeta, \frac{-1}{N}\right) B\left[\zeta(t_-), \frac{1}{N-1}\right] \\ & - \Psi\left(t, \frac{1}{N}\right) B\left(\zeta, \frac{1}{N}\right) = 0. \end{aligned}$$

Letting $N \rightarrow \infty$ in (4.5) and (4.6), we obtain

$$(4.7) \quad G_0(\zeta) = K_0(\zeta) = \frac{\alpha'_0 t + \beta'_0}{2},$$

which combined with (3.28) and (3.31) yields

$$(4.8) \quad \zeta(t) - \theta t \zeta'(t) = \pm \arccos\left(\frac{\alpha'_0 t + \beta'_0}{2}\right),$$

where the arccos function is analytically continued to $\mathbb{C} \setminus \{(0, -i\infty) \cup (t_2, \infty)\}$ so that for $t < 0$,

$$\arccos\left(\frac{\alpha'_0 t + \beta'_0}{2}\right) = \frac{1}{i} \log \left[\left(\frac{\alpha'_0 t + \beta'_0}{2}\right) + i \sqrt{1 - \left(\frac{\alpha'_0 t + \beta'_0}{2}\right)^2} \right],$$

where t_2 is given in (1.11). Recall that $\alpha'_0 < 0$ and $\beta'_0 = 2$; see the lines between (2.2) and (2.3). It is straightforward to verify by direct differentiation that the function

$$(4.9) \quad \pm \zeta(t) = \arccos\left(\frac{\alpha'_0 t + \beta'_0}{2}\right) + \alpha'_0 t^{\frac{1}{\theta}} \int_a^t \frac{\phi^{-\frac{1}{\theta}}}{\sqrt{4 - (\alpha'_0 \phi + \beta'_0)^2}} d\phi$$

is a solution to (4.8) for $t < t_2 - \sigma$, where ‘+’ sign is taken for $\theta < 2$ and ‘−’ sign is taken for $\theta > 2$. The choice of the lower limit of integration is just for the purpose of convergence of the integral. For instance, we can chose $a = 0$ if $\theta < 0$ or $\theta > 2$ and

$$(4.10) \quad a = \begin{cases} -\infty, & t < 0 \\ t_2, & 0 \leq t < t_2, \end{cases} \quad \text{if } 0 < \theta < 2.$$

With this choice, it can also be shown that $\zeta(0) = 0$.

Equating coefficients of like powers of $\frac{1}{N}$ in (4.5) and (4.6) to zero, we obtain

$$(4.11) \quad \begin{aligned} & \sum_{s \leq p, s \text{ even}} \sum_{i+m \leq s} G_i(\zeta) \binom{-p+s}{s-i-m} \sum_{l=0}^m \frac{D^l A_{p-s}(\zeta)}{l!} t^l \gamma_{l,m} \\ & + \sum_{s \leq p, s \text{ odd}} \sum_{i+m \leq s} L_i(\zeta) \binom{-p+s}{s-i-m} \sum_{l=0}^m \frac{D^l B_{p-s}(\zeta)}{l!} t^l \gamma_{l,m} \\ & - \frac{1}{2} \sum_{s \leq p} (\alpha'_s t + \beta'_s) A_{p-s}(\zeta) = 0 \end{aligned}$$

and

$$(4.12) \quad \begin{aligned} & \sum_{s \leq p, s \text{ even}} \sum_{i+m \leq s} K_i(\zeta) \binom{-p+s}{s-i-m} \sum_{l=0}^m \frac{D^l B_{p-s}(\zeta)}{l!} t^l \gamma_{l,m} \\ & + \sum_{s \leq p, s \text{ odd}} \sum_{i+m \leq s} H_i(\zeta) \binom{-p+s}{s-i-m} \sum_{l=0}^m \frac{D^l A_{p-s}(\zeta)}{l!} t^l \gamma_{l,m} \\ & - \frac{1}{2} \sum_{s \leq p} (\alpha'_s t + \beta'_s) B_{p-s}(\zeta) = 0, \end{aligned}$$

where D^j denotes the j -th derivative with respect to t , *i.e.*,

$$(4.13) \quad D^j A_l(\zeta) = \frac{d^j}{dt^j} A_l(\zeta(t)), \quad l = 0, 1, 2, \dots,$$

and $\gamma_{l,m}$ is defined by

$$(4.14) \quad \sum_{m=0}^{\infty} \frac{\gamma_{l,m}}{N^m} := \left[\left(1 + \frac{1}{N} \right)^{-\theta} - 1 \right]^l.$$

For convenience, we define

$$(4.15) \quad \begin{aligned} f_{p-1}(t) &:= \sum_{2 \leq s \leq p} \frac{\alpha'_s t + \beta'_s}{2} A_{p-s}(\zeta) \\ &\quad - \sum_{\substack{2 \leq s \leq p \\ s \text{ even}}} \left[\sum_{i+m \leq s} \binom{-p+s}{s-i-m} G_i(\zeta) \sum_{l=0}^m \frac{D^l A_{p-s}(\zeta)}{l!} \gamma_{l,m} t^l \right] \\ &\quad - \sum_{\substack{2 \leq s \leq p \\ s \text{ odd}}} \left[\sum_{i+m \leq s} \binom{-p+s}{s-i-m} L_i(\zeta) \sum_{l=0}^m \frac{D^l B_{p-s}(\zeta)}{l!} \gamma_{l,m} t^l \right] \end{aligned}$$

and

$$(4.16) \quad \begin{aligned} g_{p-1}(t) &:= \sum_{2 \leq s \leq p} \frac{\alpha'_s t + \beta'_s}{2} B_{p-s}(\zeta) \\ &\quad - \sum_{\substack{2 \leq s \leq p \\ s \text{ even}}} \left[\sum_{i+m \leq s} \binom{-p+s}{s-i-m} K_i(\zeta) \sum_{l=0}^m \frac{D^l B_{p-s}(\zeta)}{l!} \gamma_{l,m} t^l \right] \\ &\quad - \sum_{\substack{2 \leq s \leq p \\ s \text{ odd}}} \left[\sum_{i+m \leq s} \binom{-p+s}{s-i-m} H_i(\zeta) \sum_{l=0}^m \frac{D^l A_{p-s}(\zeta)}{l!} \gamma_{l,m} t^l \right] \end{aligned}$$

for $p \geq 1$. Clearly, $f_0(t) = g_0(t) = 0$. Note that $\gamma_{0,0} = 1$, $\gamma_{0,m} = 0$, $m = 1, 2, \dots$, and $\gamma_{1,0} = 0, \gamma_{1,1} = -\theta$. By (3.31)–(3.34) and (4.7), equations (4.11) and (4.12) can be rewritten as

$$(4.17) \quad [(1-p)L_0(\zeta) + L_1(\zeta)]B_{p-1}(\zeta) - \theta t L_0(\zeta) \frac{d}{dt} B_{p-1}(\zeta) = f_{p-1}(t)$$

and

$$(4.18) \quad [(1-p)H_0(\zeta) + H_1(\zeta)]A_{p-1}(\zeta) - \theta t H_0(\zeta) \frac{d}{dt} A_{p-1}(\zeta) = g_{p-1}(t),$$

where we have made use of the assumption that $\alpha_1 = \beta'_1 = 0$; cf. (2.2) and (2.3). It follows from (3.31), (3.32) and (4.7) that

$$(4.19) \quad H_0(\zeta) = -\sqrt{\frac{4 - (\alpha'_0 t + \beta'_0)^2}{4}}, \quad \frac{dH_0(\zeta)}{dt} = -\frac{\alpha'_0}{2} \frac{G_0(\zeta)}{H_0(\zeta)}.$$

Differentiating (4.8), we get

$$(4.20) \quad (1 - \theta)\zeta'(t) - \theta t \zeta''(t) = \frac{\alpha'_0}{2H_0(\zeta)}.$$

Combining (3.28), (3.32), (3.34), (4.19) and (4.20), we obtain

$$(4.21) \quad H_1(\zeta) = -\frac{\theta t}{2} \frac{d}{dt} H_0(\zeta) - \frac{1}{2} \left[1 - \theta t \frac{\zeta'}{\zeta} \right] H_0(\zeta)$$

and

$$(4.22) \quad L_1(\zeta) = -\frac{\theta t}{2} \frac{d}{dt} L_0(\zeta) - \frac{1}{2} \left[1 - \theta t \frac{\zeta'}{\zeta} \right] L_0(\zeta).$$

Substituting (4.21) and (4.22) into (4.17) and (4.18) and noting that $L_0(\zeta) = -H_0(\zeta)$, we have

$$(4.23) \quad \begin{cases} \frac{d}{dt} [t^{\frac{p-1}{\theta}} \Lambda(t) B_{p-1}] = \frac{t^{\frac{p-1}{\theta}-1} \Lambda(t)}{\theta H_0(\zeta)} f_{p-1}(t), \\ \frac{d}{dt} [t^{\frac{p-1}{\theta}} \Lambda(t) A_{p-1}] = -\frac{t^{\frac{p-1}{\theta}-1} \Lambda(t)}{\theta H_0(\zeta)} g_{p-1}(t), \end{cases}$$

where

$$(4.24) \quad \Lambda(t) := t^{\frac{1}{2\theta}} \left[\frac{-H_0(\zeta)}{\zeta} \right]^{\frac{1}{2}}.$$

For $p = 1$, since $f_0 = g_0 = 0$, $\Lambda A_0(\zeta)$ and $\Lambda B_0(\zeta)$ are constants, we set

$$(4.25) \quad \Lambda A_0(\zeta) = 1, \quad \Lambda B_0 = 0.$$

For $p > 1$ and $0 < \theta < 2$

$$(4.26) \quad t^{\frac{p-1}{\theta}} \Lambda(t) B_{p-1}(\zeta) = \int_0^t \frac{s^{\frac{p-1}{\theta}-1} \Lambda(s)}{\theta H_0(\zeta(s))} f_{p-1}(s) ds, \quad t < t_2 - \sigma$$

and

$$(4.27) \quad t^{\frac{p-1}{\theta}} \Lambda(t) A_{p-1}(\zeta) = - \int_0^t \frac{s^{\frac{p-1}{\theta}-1} \Lambda(s)}{\theta H_0(\zeta(s))} g_{p-1}(s) ds, \quad t < t_2 - \sigma.$$

Therefore, for each $p > 1$, $A_p(t)$ and $B_p(t)$ can be determined successively from their predecessors $A_0(t)$, $B_0(t)$, \dots , $A_{p-1}(t)$ and $B_{p-1}(t)$. If, in addition, we have the assumption $\alpha'_{2s+1} = \beta'_{2s+1} = 0$ for $s = 0, 1, \dots$, we can choose the solutions such that $A_{2s+1} = B_{2s} = 0$.

5. BOUNDS FOR COEFFICIENTS

Lemma 5.1. *Let $\zeta(t)$ be given as in (4.9) and (4.10), and let $G(\zeta, \frac{1}{N})$, $H(\zeta, \frac{1}{N})$, $L(\zeta, \frac{1}{N})$ and $K(\zeta, \frac{1}{N})$ be given as in Lemma 3.1. Then there exists a constant C_s independent of t such that*

$$(5.1) \quad |G_s(\zeta)| \leq C_s(|t| + 1), \quad |H_s(\zeta)| \leq C_s(|t| + 1),$$

$$(5.2) \quad |K_s(\zeta)| \leq C_s(|t| + 1), \quad |L_s(\zeta)| \leq C_s(|t| + 1)$$

for all $t \leq t_2 - \sigma$, $\sigma > 0$.

Proof. Note that the function $\zeta^2(t)$ in (4.9) is C^∞ in $(-\infty, t_2 - \sigma)$, and that each u_k , $k = 0, 1, \dots$, given in (3.27), is continuous at $t = 0$. Hence the expansions in (3.4) and (3.5) hold uniformly with respect to t in any compact subinterval of $(-\infty, t_2)$, and the coefficients $G_s(\zeta)$ and $H_s(\zeta)$ in these expansions are bounded for all finite t away from t_2 . Thus, to prove the two estimates in (5.1), it suffices to show that the following two limits exist:

$$(5.3) \quad \lim_{t \rightarrow -\infty} \frac{G_s(\zeta)}{t}, \quad \lim_{t \rightarrow -\infty} \frac{H_s(\zeta)}{t}.$$

Replacing Z_ν by J_ν and Y_ν and choosing positive sign in (3.2), we obtain

$$(5.4) \quad J_\nu \{(N+1)\zeta(t_+)\} = J_\nu(N\zeta)G\left(\zeta, \frac{1}{N}\right) + J_{\nu+1}(N\zeta)H\left(\zeta, \frac{1}{N}\right),$$

and

$$(5.5) \quad Y_\nu \{(N+1)\zeta(t_+)\} = Y_\nu(N\zeta)G\left(\zeta, \frac{1}{N}\right) + Y_{\nu+1}(N\zeta)H\left(\zeta, \frac{1}{N}\right).$$

Upon solving the last two equations for $G(\zeta, \frac{1}{N})$ and $H(\zeta, \frac{1}{N})$, we have

$$(5.6) \quad G\left(\zeta, \frac{1}{N}\right) = \frac{\pi N \zeta}{2} \{ J_{\nu+1}(N\zeta)Y_\nu[(N+1)\zeta(t_+)] - Y_{\nu+1}(N\zeta)J_\nu[(N+1)\zeta(t_+)] \}$$

and

$$(5.7) \quad H\left(\zeta, \frac{1}{N}\right) = \frac{\pi N \zeta}{2} \{ Y_\nu(N\zeta)J_\nu[(N+1)\zeta(t_+)] - J_\nu(N\zeta)Y_\nu[(N+1)\zeta(t_+)] \},$$

where we have made use of the identity

$$J_{\nu+1}(x)Y_\nu(x) - J_\nu(x)Y_{\nu+1}(x) = \frac{2}{\pi x};$$

see [15, p. 244] and [16, eq. (10.5.2)]. Recall the relations between the Bessel functions and modified Bessel functions [15, pp. 60 & 251]

$$(5.8) \quad J_\nu(ix) = e^{\nu\pi i/2} I_\nu(x), \quad Y_\nu(ix) = e^{(\nu+1)\pi i/2} I_\nu(x) - \frac{2}{\pi} e^{-\nu\pi i/2} K_\nu(x),$$

and let

$$(5.9) \quad \xi = -iN\zeta(t), \quad \xi' = -i(N+1)\zeta(t_+).$$

Then, (5.6) can be written as

$$(5.10) \quad G\left(\zeta, \frac{1}{N}\right) = \xi [I_\nu(\xi')K_{\nu+1}(\xi) + K_\nu(\xi')I_{\nu+1}(\xi)].$$

Recall the asymptotic expansions of the modified Bessel functions for large argument

$$(5.11) \quad I_\nu(x) \sim \frac{e^x}{(2\pi x)^{1/2}} \sum_{s=0}^{\infty} (-1)^s \frac{\mu_s}{x^s}$$

and

$$(5.12) \quad K_\nu(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \sum_{s=0}^{\infty} \frac{\mu_s}{x^s},$$

where $\mu_0 = 1$; see [15, pp. 250-251, 238]. A combination of (5.10)-(5.12) gives

$$(5.13) \quad G\left(\zeta, \frac{1}{N}\right) \sim \frac{1}{2} \left(\frac{\xi}{\xi'}\right)^{\frac{1}{2}} \left[e^{\xi' - \xi} \sum_{s=0}^{\infty} (-1)^s \frac{\mu_s}{\xi^s} \left(\frac{\xi}{\xi'}\right)^s \sum_{l=0}^{\infty} \frac{\tilde{\mu}_l}{\xi'^l} + e^{\xi - \xi'} \sum_{s=0}^{\infty} (-1)^s \frac{\tilde{\mu}_s}{\xi^s} \sum_{l=0}^{\infty} \frac{\mu_l}{\xi'^l} \left(\frac{\xi}{\xi'}\right)^l \right],$$

where $\tilde{\mu}_0 = 1$ and

$$\tilde{\mu}_s(\nu) = \mu_s(\nu + 1), \quad s \geq 1.$$

It follows from (4.9) and (4.10) that as $t \rightarrow -\infty$

$$(5.14) \quad \zeta^{\pm 1}(t) = \mathcal{O}\left(|t|^{\pm \frac{1}{\theta}}\right), \quad \theta > 2,$$

and

$$(5.15) \quad \zeta^{\pm 1}(t) = \mathcal{O}\left(\log^{\pm 1} |t|\right), \quad 0 < \theta < 2 \text{ or } \theta < 0.$$

Using (4.9), (4.10) and (5.14), a straightforward calculation gives

$$(5.16) \quad \begin{aligned} \xi' - \xi = \log & \frac{\alpha'_0 t + \beta'_0 + \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4}}{2} \\ & + \frac{\theta \alpha'_0 t}{N} \int_0^1 \frac{(s-1)(1 + \frac{s}{N})^{-\theta-1}}{\sqrt{(\alpha'_0 t(1 + \frac{s}{N})^{-\theta} + \beta'_0)^2 - 4}} ds := \sum_{j=0}^{\infty} \frac{\delta_j(t)}{N^j}, \end{aligned}$$

where $\delta_0(t) = \log[(\alpha'_0 t + \beta'_0 + \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4})/2]$ and

$$(5.17) \quad \delta_j(t) = \frac{(-1)^j \theta}{j(j+1)} + \mathcal{O}_j(|t|^{-1}) \quad \text{as } t \rightarrow -\infty.$$

Note: in (5.17) we have used a subscript “ j ” to indicate that each \mathcal{O} -term depends on j , $j = 1, 2, \dots$. Hence,

$$(5.18) \quad \frac{\xi'}{\xi} = 1 + i \sum_{j=1}^{\infty} \frac{\delta_{j-1}(t) \zeta^{-1}(t)}{N^j}.$$

For each $j \geq 1$, the coefficient of N^{-j} is $\mathcal{O}(\log^{-1} |t|)$ for large negative t . We define

$$\begin{aligned} \exp \left[\sum_{j=1}^{\infty} \frac{1}{N^j} \left(\delta_j(t) - \frac{(-1)^j \theta}{j(j+1)} \right) \right] &:= \sum_{j=0}^{\infty} \frac{a_j(t)}{N^j}, \\ \exp \left[\sum_{j=1}^{\infty} \frac{1}{N^j} \left(-\delta_j(t) - \frac{(-1)^j \theta}{j(j+1)} \right) \right] &:= \sum_{j=0}^{\infty} \frac{b_j(t)}{N^j} \end{aligned}$$

and

$$\left(\frac{\xi}{\xi'}\right)^l := \sum_{j=0}^{\infty} \frac{c_{l,j}(t)}{N^j}.$$

It is readily verified that $a_0(t) = b_0(t) = c_{l,0}(t) = 1$, $a_j(t) = \mathcal{O}_j(|t|^{-1})$ for each $j \geq 1$ and each $l \geq 0$, and other $b_j(t)$, $c_{l,j}(t)$ are also bounded for $t \leq t_2 - \sigma$. With these notations we have from (5.13) that

$$\begin{aligned} 2G\left(\zeta, \frac{1}{N}\right) &\sim \exp\left[\delta_0(t) + \sum_{j=1}^{\infty} \frac{(-1)^j \theta}{j(j+1)} \cdot \frac{1}{N^j}\right] \cdot \sum_{j=0}^{\infty} \frac{a_j(t)}{N^j} \\ &\quad \times \sum_{s=0}^{\infty} \frac{\tilde{\mu}_s}{N^s} (\mathrm{i}\zeta^{-1}(t))^s \cdot \sum_{l=0}^{\infty} \frac{(-1)^l \mu_l}{N^l} (\mathrm{i}\zeta^{-1}(t))^l \sum_{m=0}^{\infty} \frac{c_{l+\frac{1}{2},m}(t)}{N^m} \\ &\quad + \exp\left[-\delta_0(t) + \sum_{j=1}^{\infty} \frac{(-1)^j \theta}{j(j+1)} \frac{1}{N^j}\right] \cdot \sum_{j=0}^{\infty} \frac{b_j(t)}{N^j} \\ &\quad \times \sum_{s=0}^{\infty} (-1)^s \frac{\tilde{\mu}_s}{N^s} (\mathrm{i}\zeta^{-1}(t))^s \cdot \sum_{l=0}^{\infty} \frac{\mu_l}{N^l} (\mathrm{i}\zeta^{-1}(t))^l \sum_{m=0}^{\infty} \frac{c_{l+\frac{1}{2},m}(t)}{N^m}, \end{aligned}$$

which in turn gives

$$\begin{aligned} 2G\left(\zeta, \frac{1}{N}\right) &\sim e^{\theta} \left(1 + \frac{1}{N}\right)^{-(N+1)\theta} \left[\frac{\alpha'_0 t + \beta'_0 + \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4}}{2} \cdot \sum_{p=0}^{\infty} \frac{w_{1,p}(t)}{N^p} \right] \\ &\quad + e^{\theta} \left(1 + \frac{1}{N}\right)^{-(N+1)\theta} \left[\frac{\alpha'_0 t + \beta'_0 - \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4}}{2} \cdot \sum_{p=0}^{\infty} \frac{w_{2,p}(t)}{N^p} \right], \end{aligned}$$

where $w_{1,0}(t) = w_{2,0}(t) = 1$ and for $p \geq 1$

$$w_{1,p}(t) = \sum_{j+s+l+m=p} (-1)^l a_j(t) \tilde{\mu}_s \mu_l [\mathrm{i}\zeta^{-1}(t)]^{s+l} c_{l+\frac{1}{2},m}(t),$$

$$w_{2,p}(t) = \sum_{j+s+l+m=p} (-1)^s b_j(t) \tilde{\mu}_s \mu_l [\mathrm{i}\zeta^{-1}(t)]^{s+l} c_{l+\frac{1}{2},m}(t).$$

Note that for each $p \geq 1$, we have $w_{1,p} = \mathcal{O}(1)$ and $w_{2,p} = \mathcal{O}(1)$. If we set

$$\tau_p(t) = \frac{\alpha'_0 t + \beta'_0 + \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4}}{2(\alpha'_0 t + \beta'_0)} w_{1,p}(t) + \frac{\alpha'_0 t + \beta'_0 - \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4}}{2(\alpha'_0 t + \beta'_0)} w_{2,p}(t),$$

then we further obtain

$$(5.19) \quad G\left(\zeta, \frac{1}{N}\right) \sim \frac{\alpha'_0 t + \beta'_0}{2} e^{\theta} \left(1 + \frac{1}{N}\right)^{-(N+1)\theta} \sum_{p=0}^{\infty} \frac{\tau_p(t)}{N^p},$$

where $\tau_0(t) = 1$ and $\tau_p(t) = \mathcal{O}(1)$ for each $p \geq 1$. Let

$$(5.20) \quad e^\theta \left(1 + \frac{1}{N}\right)^{-(N+1)\theta} = \exp \left[\theta - \theta(N+1) \log \left(1 + \frac{1}{N}\right) \right] := \sum_{s=0}^{\infty} \frac{\omega_s}{N^s}.$$

A combination of (5.19) and (5.20) yields

$$(5.21) \quad G\left(\zeta, \frac{1}{N}\right) \sim \frac{\alpha'_0 t + \beta'_0}{2} \sum_{s=0}^{\infty} \frac{1}{N^s} \sum_{j=0}^s \omega_{s-j} \tau_j(t),$$

and a comparison of (3.4) and (5.21) gives

$$(5.22) \quad G_s(\zeta) = \frac{\alpha'_0 t + \beta'_0}{2} [\omega_s + \mathcal{O}(1)].$$

The existence of the first limit in (5.3) is thus proved. The existence of the second limit in (5.3) can be shown in a similar manner, and we have thus established the two estimates in (5.1). The proofs of the two inequalities in (5.2) are very similar to the ones given above, and hence will not be repeated here. This completes the proof of Lemma 5.1. \square

Using the estimates proved in Lemma 5.1, we can provide some bounds for the coefficient functions $A_s(\zeta)$ and $B_s(\zeta)$ in (4.1). We state the result as follows.

Lemma 5.2. *For $s, j = 0, 1, 2, \dots$ and $t < 0$, there exists a constant $M_{s,j}$ such that*

$$(5.23) \quad \left| t^{j+\frac{1}{2\theta}} (-H_0/\zeta)^{\frac{1}{2}} D^j A_s(\zeta) \right| \leq M_{s,j} (1 + |t|^{-\frac{s}{\theta}})$$

and

$$(5.24) \quad \left| t^{j+\frac{1}{2\theta}} (-H_0/\zeta)^{\frac{1}{2}} D^j B_s(\zeta) \right| \leq M_{s,j} (1 + |t|^{-\frac{s}{\theta}}),$$

where D^j denotes the j -th derivative with respect to t .

Proof. Our approach is based on mathematical induction. Since $\zeta(t)$, $G_s(\zeta)$, $H_s(\zeta)$, $L_s(\zeta)$ and $K_s(\zeta)$ are all analytic functions in the left half plane $\operatorname{Re} t < 0$, and since the estimates in Lemma 5.1 also hold for complex t in this half plane, an application of Cauchy's integral formula shows that for $s, j = 0, 1, 2, \dots$ and $t < 0$, there exist constants \tilde{N}_j and $N_{s,j}$ such that

$$(5.25) \quad |t^j D^j \zeta| \leq \tilde{N}_j (1 + |\zeta|),$$

$$(5.26) \quad |t^j D^j G_s(\zeta)| \leq N_{s,j} (1 + |t|),$$

$$(5.27) \quad |t^j D^j H_s(\zeta)| \leq N_{s,j} (1 + |t|),$$

$$(5.28) \quad |t^j D^j K_s(\zeta)| \leq N_{s,j} (1 + |t|),$$

$$(5.29) \quad |t^j D^j L_s(\zeta)| \leq N_{s,j} (1 + |t|).$$

When $s = 0$, the estimates in (5.23) and (5.24) follow from (4.25) for $j = 0$. This, together with (4.17) and (4.18), implies that (5.23) and (5.24) hold for $j = 1$. Furthermore, by differentiating (4.17) and (4.18) j times with respect to t , one can show that (5.23) and (5.24) also hold for $j \geq 2$. Now assume these estimates hold for $s = 0, 1, \dots, p-2$, $p \geq 2$. We shall show that they are valid for $s = p-1$. Using (5.25)–(5.29), it can be shown from (4.15) and (4.16) that

$$(5.30) \quad |t^{\frac{1}{2\theta}}(-H_0/\zeta)^{\frac{1}{2}}f_{p-1}(t)| \leq C_{p,0}(1+|t|)(1+|t|^{-\frac{p-2}{\theta}})$$

and

$$(5.31) \quad |t^{\frac{1}{2\theta}}(-H_0/\zeta)^{\frac{1}{2}}g_{p-1}(t)| \leq C_{p,0}(1+|t|)(1+|t|^{-\frac{p-2}{\theta}}),$$

where $C_{p,0}$ is a positive constant. Combining the last two inequalities with (4.27) leads to

$$(5.32) \quad \begin{aligned} |t^{\frac{1}{2\theta}}(-H_0/\zeta)^{\frac{1}{2}}A_{p-1}(\zeta)| &= |t|^{-\frac{p-1}{\theta}} \left| C + \int_{-1}^t \frac{s^{\frac{p-1}{\theta}-1}}{\theta H_0(\zeta(s))} \Lambda(s) f_{p-1}(s) ds \right| \\ &\leq |t|^{-\frac{p-1}{\theta}} \left(C + \tilde{C}_{p,0} \left| \int_{-1}^t |s|^{\frac{p-1}{\theta}-1} (1 + |s|^{-\frac{p-2}{\theta}}) ds \right| \right) \\ &\leq |t|^{-\frac{p-1}{\theta}} \left(C + \tilde{C}_{p,0} \int_1^{-t} (\tau^{\frac{p-1}{\theta}-1} + \tau^{\frac{1}{\theta}-1}) d\tau \right) \\ &\leq M_{p-1,0}(1 + |t|^{-\frac{p-1}{\theta}}). \end{aligned}$$

Here we have made use of (4.19). In a similar manner, we have

$$(5.33) \quad |t^{\frac{1}{2\theta}}(-H_0/\zeta)^{\frac{1}{2}}B_{p-1}(\zeta)| \leq M_{p-1,0}(1 + |t|^{-\frac{p-1}{\theta}}).$$

Hence, (5.23) and (5.24) hold for $s = p-1$ and $j = 0$. Using (4.17) and (4.18), one can also show that (5.23) and (5.24) are valid for $s = p-1$ and $j = 1, 2, \dots$. This completes the proof of Lemma 5.2 by induction. \square

6. ASYMPTOTIC NATURE OF THE EXPANSION

Since x is fixed in the recurrence relation (1.7), we let $W_\nu(x) := Y_\nu(x) - iJ_\nu(x)$ and choose

$$Z_\nu(N\zeta) = x^{\frac{1}{2\theta}} J_\nu(N\zeta) \quad \text{and} \quad \bar{Z}_\nu(N\zeta) = x^{\frac{1}{2\theta}} W_\nu(N\zeta)$$

in (4.1) to yield two linearly independent solutions

$$(6.1) \quad \begin{aligned} P_n(N^\theta t) &= \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} N^{\frac{1}{2}} \left[J_\nu(N\zeta) \sum_{s=0}^p \frac{\tilde{A}_s(\zeta)}{N^s} \right. \\ &\quad \left. + J_{\nu+1}(N\zeta) \sum_{s=0}^p \frac{\tilde{B}_s(\zeta)}{N^s} + \varepsilon_n^p(N, t) \right] \end{aligned}$$

and

$$(6.2) \quad Q_n(N^\theta t) = \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} N^{\frac{1}{2}} \left[W_\nu(N\zeta) \sum_{s=0}^p \frac{\tilde{A}_s(\zeta)}{N^s} + W_{\nu+1}(N\zeta) \sum_{s=0}^p \frac{\tilde{B}_s(\zeta)}{N^s} + \delta_n^p(N, t) \right],$$

where $\tilde{A}_s(\zeta) = t^{\frac{1}{2\theta}} [-H_0(\zeta)/\zeta]^{\frac{1}{2}} A_s(\zeta)$, $\tilde{B}_s(\zeta) = t^{\frac{1}{2\theta}} [-H_0(\zeta)/\zeta]^{\frac{1}{2}} B_s(\zeta)$, and

$$(6.3) \quad \tilde{A}_0(\zeta) = 1, \quad \tilde{B}_0(\zeta) = 0.$$

Here we have made use of (4.19), (4.24) and (4.25). From (4.9) and (4.10), one can get the behavior of $\zeta(t)$ as $t \rightarrow -\infty$; cf. (5.14) and (5.15). With this, it follows from Cauchy's integral formula that there is a constant C_k such that

$$|t^k D^k \Lambda(t)| \leq C_k |\Lambda(t)|, \quad t < 0.$$

Applying Leibniz's rule to the products $\tilde{A}_s(\zeta) = \Lambda(t) A_s(\zeta)$ and $\tilde{B}_s(\zeta) = \Lambda(t) B_s(\zeta)$, the estimates in (5.23) and (5.24) give

$$(6.4) \quad |t^j D^j \tilde{A}_s(\zeta)| \leq \tilde{M}_{s,j} (1 + |t|^{-\frac{s}{\theta}})$$

and

$$(6.5) \quad |t^j D^j \tilde{B}_s(\zeta)| \leq \tilde{M}_{s,j} (1 + |t|^{-\frac{s}{\theta}}),$$

where $\tilde{M}_{s,j}$ is some other constant. It readily follows from (6.3) that $\tilde{A}_0(\zeta)$ and $\tilde{B}_0(\zeta)$ are bounded for $t < t_2 - \sigma$. Assume that $\tilde{A}_s(\zeta)$ and $\tilde{B}_s(\zeta)$ are bounded for $t < t_2 - \sigma$ and $s \leq p$. Using an argument similar to that given for (5.30)-(5.33), we have from (4.15) and (4.16) that

$$|t^{\frac{1}{2\theta}} (-H_0/\zeta)^{\frac{1}{2}} f_{p+1}(t)| \leq C_{p+1,0} (1 + |t|),$$

$$|t^{\frac{1}{2\theta}} (-H_0/\zeta)^{\frac{1}{2}} g_{p+1}(t)| \leq C_{p+1,0} (1 + |t|).$$

As a consequence, it can be shown that (5.23) and (5.24) can be improved to read

$$|t^{j+\frac{1}{2\theta}} (-H_0/\zeta)^{\frac{1}{2}} D^j A_s(\zeta)| \leq M_{s,j}$$

and

$$|t^{j+\frac{1}{2\theta}} (-H_0/\zeta)^{\frac{1}{2}} D^j B_s(\zeta)| \leq M_{s,j}$$

for $t < 0$ and $s, j \geq 0$. In particular, these two estimates hold for $j = 0$ and $s = p + 1$; that is, $\tilde{A}_{p+1}(\zeta)$ and $\tilde{B}_{p+1}(\zeta)$ are bounded for $t < 0$. By the continuity of $\zeta(t)$, $G_s(\zeta)$ and $H_s(\zeta)$ we conclude that $\tilde{A}_{p+1}(\zeta)$ and $\tilde{B}_{p+1}(\zeta)$ are also bounded for $t < t_2 - \sigma$, thus completing the induction argument.

Now we shall show that (1.7) has two linearly independent solutions $P_n(x)$ and $Q_n(x)$ with (6.1) and (6.2) being their uniform asymptotic expansions for $t < t_2 - \sigma$. We need to show that (2.20) and (2.21) hold for $t < t_2 - \sigma$.

For convenience, we introduce the notations

$$(6.6) \quad A_p\left(\zeta, \frac{1}{N}\right) := \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2}\right)^{\frac{1}{4}} N^{\frac{1}{2}} \sum_{s=0}^p \frac{\tilde{A}_s(\zeta)}{N^s},$$

$$(6.7) \quad B_p\left(\zeta, \frac{1}{N}\right) := \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2}\right)^{\frac{1}{4}} N^{\frac{1}{2}} \sum_{s=0}^p \frac{\tilde{B}_s(\zeta)}{N^s},$$

$$(6.8) \quad r_n^p(x) := J_\nu(N\zeta)A_p\left(\zeta, \frac{1}{N}\right) + J_{\nu+1}(N\zeta)B_p\left(\zeta, \frac{1}{N}\right)$$

and

$$(6.9) \quad s_n^p(x) := W_\nu(N\zeta)A_p\left(\zeta, \frac{1}{N}\right) + W_{\nu+1}(N\zeta)B_p\left(\zeta, \frac{1}{N}\right).$$

By Lemma 3.1 we have

$$(6.10) \quad r_{n+1}^p(x) - (A_n x + B_n)r_n^p(x) + r_{n-1}^p(x) = \frac{R_n^p(x)}{N^{p+\frac{3}{2}}},$$

where $x = N^\theta t$ and the inhomogeneous term is given by

$$(6.11) \quad \frac{R_n^p(x)}{N^{p+\frac{3}{2}}} = J_\nu(N\zeta)F_{1,n}(x) + J_{\nu+1}(N\zeta)F_{2,n}(x)$$

with

$$(6.12) \quad \begin{aligned} F_{1,n}(x) = & G\left(\zeta, \frac{1}{N}\right)A_p\left(\zeta(t_+), \frac{1}{N+1}\right) + G\left(\zeta, -\frac{1}{N}\right)A_p\left(\zeta(t_-), \frac{1}{N-1}\right) \\ & + L\left(\zeta, \frac{1}{N}\right)B_p\left(\zeta(t_+), \frac{1}{N+1}\right) - L\left(\zeta, -\frac{1}{N}\right)B_p\left(\zeta(t_-), \frac{1}{N-1}\right) \\ & - \Psi\left(t, \frac{1}{N}\right)A_p\left(\zeta, \frac{1}{N}\right) \end{aligned}$$

and

$$(6.13) \quad \begin{aligned} F_{2,n}(x) = & K\left(\zeta, \frac{1}{N}\right)B_p\left(\zeta(t_+), \frac{1}{N+1}\right) + K\left(\zeta, -\frac{1}{N}\right)B_p\left(\zeta(t_-), \frac{1}{N-1}\right) \\ & + H\left(\zeta, \frac{1}{N}\right)A_p\left(\zeta(t_+), \frac{1}{N+1}\right) - H\left(\zeta, -\frac{1}{N}\right)A_p\left(\zeta(t_-), \frac{1}{N-1}\right) \\ & - \Psi\left(t, \frac{1}{N}\right)B_p\left(\zeta, \frac{1}{N}\right); \end{aligned}$$

cf. (4.5) and (4.6). Note that the series

$$A\left(\zeta, \frac{1}{N}\right) := \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} \quad \text{and} \quad B\left(\zeta, \frac{1}{N}\right) := \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}$$

in (4.2) are formal solutions of (4.5) and (4.6). Since $A_p(\zeta, \frac{1}{N})$ and $B_p(\zeta, \frac{1}{N})$ can be rewritten as

$$(6.14) \quad A_p\left(\zeta, \frac{1}{N}\right) := x^{\frac{1}{2\theta}} \sum_{s=0}^{\infty} \frac{A_s^*(\zeta)}{N^s} \quad \text{and} \quad B_p\left(\zeta, \frac{1}{N}\right) := x^{\frac{1}{2\theta}} \sum_{s=0}^{\infty} \frac{B_s^*(\zeta)}{N^s}$$

with $A_s^*(\zeta) = A_s(\zeta)$, $B_s^*(\zeta) = B_s(\zeta)$, for $s \leq p$ and $A_s^*(\zeta) = 0$, $B_s^*(\zeta) = 0$ for $s > p$, terms with powers of $1/N$ less than or equal to $p+1$ in the expansions $F_{1,n}(x)$ and $F_{2,n}(x)$ all vanish, *i.e.*, if we write $(N^\theta t)^{-\frac{1}{2\theta}} F_{1,n}(x) \sim \sum_{s=0}^{\infty} f_{1,s}(t) N^{-s}$ and $(N^\theta t)^{-\frac{1}{2\theta}} F_{2,n}(x) \sim \sum_{s=0}^{\infty} f_{2,s}(t) N^{-s}$, then $f_{1,s}(t) = f_{2,s}(t) = 0$ for $0 \leq s \leq p+1$. (For $s = 0, 1, \dots, p$, all we need are recurrence relations obtained from (4.5) and (4.6); for $s = p+1$, there are extra terms and we have to show that these terms indeed all cancel out.) Using (5.6)–(5.12), it can be shown that there is a constant C such that

$$\left| G\left(\zeta, \frac{1}{N}\right) \right| \leq C(1 + |t|), \quad \left| H\left(\zeta, \frac{1}{N}\right) \right| \leq C(1 + |t|)$$

and

$$\left| K\left(\zeta, \frac{1}{N}\right) \right| \leq C(1 + |t|), \quad \left| L\left(\zeta, \frac{1}{N}\right) \right| \leq C(1 + |t|)$$

for all $t < t_2 - \sigma$. Recall that $\tilde{A}_s(\zeta)$ and $\tilde{B}_s(\zeta)$ are bounded in $t < t_2 - \sigma$ for $s \leq p$. Hence by Lemma 5.1, it can be proved that there exists a constant C_p such that

$$(6.15) \quad |t^{\frac{1}{2\theta}} (-H_0/\zeta)^{\frac{1}{2}} (N^\theta t)^{-\frac{1}{2\theta}} F_{1,n}(N^\theta t)| \leq C_p(1 + |t|)/N^{p+2}$$

and

$$(6.16) \quad |t^{\frac{1}{2\theta}} (-H_0/\zeta)^{\frac{1}{2}} (N^\theta t)^{-\frac{1}{2\theta}} F_{2,n}(N^\theta t)| \leq C_p(1 + |t|)/N^{p+2}$$

for all $t < t_2 - \sigma$. Then, the last two inequalities, together with (6.11), lead to

$$(6.17) \quad |R_n^p(N^\theta t)| \leq \tilde{C}_p(1 + |t|) \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|J_\nu(N\zeta)| + |J_{\nu+1}(N\zeta)|]$$

for $t < t_2 - \sigma$ and for some positive constant \tilde{C}_p . Similarly, we have

$$(6.18) \quad s_{n+1}^p(x) - (A_n x + B_n) s_n^p(x) + s_{n-1}^p(x) = \frac{S_n^p(x)}{N^{p+\frac{3}{2}}}$$

with

$$(6.19) \quad |S_n^p(N^\theta t)| \leq \tilde{C}_p(1 + |t|) \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|W_\nu(N\zeta)| + |W_{\nu+1}(N\zeta)|].$$

We now establish the existence of two solutions $P_n(x)$ and $Q_n(x)$ of (1.7) satisfying

$$(6.20) \quad P_n(x) \sim r_n^0(x) \quad \text{and} \quad Q_n(x) \sim s_n^0(x)$$

as $n \rightarrow \infty$ for $x = N^\theta t$ with any fixed $t < t_2 - \sigma$. In view of (5.8), by using (5.11) and (5.12) it is easily verified that

$$(6.21) \quad r_n^0(x) \sim \sqrt{\frac{1}{2\pi}} e^{\nu\pi i/2} \left(\frac{4}{(\alpha'_0 t + \beta'_0)^2 - 4} \right)^{\frac{1}{4}} e^{-iN\zeta}$$

and

$$(6.22) \quad s_n^0(x) \sim -\sqrt{\frac{2}{\pi}} e^{-\nu\pi i/2} \left(\frac{4}{(\alpha'_0 t + \beta'_0)^2 - 4} \right)^{\frac{1}{4}} e^{iN\zeta}$$

as $n \rightarrow \infty$ for any fixed $t < 0$. From (3.24), (3.27), (3.28) and (4.8), it follows that

$$(N+1)\zeta(t_+) - N\zeta(t) \sim \frac{1}{i} \log \left[\left(\frac{\alpha' t + \beta'_0}{2} \right) + i \sqrt{1 - \left(\frac{\alpha' t + \beta'_0}{2} \right)^2} \right]$$

as $n \rightarrow \infty$. If $P_n(x)$ and $Q_n(x)$ are two solutions of (1.7) satisfying (6.20), then we have from (6.20)–(6.22)

$$(6.23) \quad P_{n+1}(x)Q_n(x) - P_n(x)Q_{n+1}(x) = P_{n+2}(x)Q_{n+1}(x) - P_{n+1}(x)Q_{n+2}(x)$$

and

$$(6.24) \quad \begin{aligned} & P_{n+1}(x)Q_n(x) - P_n(x)Q_{n+1}(x) \\ &= \lim_{m \rightarrow \infty} [r_{m+1}^0(x)s_m^0(x) - r_m^0(x)s_{m+1}^0(x)] = \frac{2}{\pi} \end{aligned}$$

for fixed $t < 0$. Examining the behavior of the Bessel functions, one can show that (6.24) also holds for fixed $t \geq 0$. This, in particular, demonstrates that $P_n(x)$ and $Q_n(x)$ are two linearly independent solutions.

Now define

$$(6.25) \quad \varepsilon_n^p(x) := P_n(x) - r_n^p(x) \quad \text{and} \quad \delta_n^p(x) := Q_n(x) - s_n^p(x).$$

We first show that the existence of $Q_n(x)$ to (1.7) satisfying (6.20) is equivalent to the existence of $\delta_n^p(x)$ to the summation formula

$$(6.26) \quad \begin{aligned} \delta_n^p(x) &= \sum_{j=n+1}^{\infty} \frac{[s_n^p(x)r_j^p(x) - r_n^p(x)s_j^p(x)]S_j^p(x)}{[s_{n+1}^p(x)r_n^p(x) - s_n^p(x)r_{n+1}^p(x)](j + \tau_0)^{p+\frac{3}{2}}} \\ &\quad + \sum_{j=n+1}^{\infty} \frac{[s_n^p(x)R_j^p(x) - r_n^p(x)S_j^p(x)]\delta_j^p(x)}{[s_{n+1}^p(x)r_n^p(x) - s_n^p(x)r_{n+1}^p(x)](j + \tau_0)^{p+\frac{3}{2}}}. \end{aligned}$$

From (1.7) and (6.18), we obtain

$$(6.27) \quad \delta_{n+1}^p(x) - (A_n x + B_n)\delta_n^p(x) + \delta_{n-1}^p(x) = -\frac{S_n^p(x)}{N^{p+\frac{3}{2}}}.$$

Coupling (6.10) and (6.27) gives

$$(6.28) \quad \begin{aligned} r_n^p(x)\delta_{n+1}^p(x) - r_{n+1}^p(x)\delta_n^p(x) &= r_{n+1}^p(x)\delta_{n+2}^p(x) - r_{n+2}^p(x)\delta_{n+1}^p(x) \\ &+ \frac{S_{n+1}^p(x)r_{n+1}^p(x) + R_{n+1}^p(x)\delta_{n+1}^p(x)}{(N+1)^{p+3/2}}, \end{aligned}$$

which leads to

$$(6.29) \quad \begin{aligned} r_n^p(x)\delta_{n+1}^p(x) - r_{n+1}^p(x)\delta_n^p(x) &= r_{m+1}^p(x)\delta_{m+2}^p(x) - r_{m+2}^p(x)\delta_{m+1}^p(x) \\ &+ \sum_{j=n+1}^{m+1} \frac{S_j^p(x)r_j^p(x) + R_j^p(x)\delta_j^p(x)}{(j+\tau_0)^{p+3/2}}. \end{aligned}$$

If $Q_n(x)$ satisfies (6.20), then $\delta_n^0(x) = o(s_n^0(x))$ as $n \rightarrow \infty$. Since $\delta_n^p(x) = \delta_n^0(x) + s_n^0(x) - s_n^p(x)$ by (6.25), as well as $\tilde{A}_s(\zeta)$ and $\tilde{B}_s(\zeta)$ are bounded for $s \geq 0$, on account of (6.8) we also have $\delta_n^p(x) = o(s_n^0(x))$ as $n \rightarrow \infty$. In view of (6.21) and (6.22), we have

$$(6.30) \quad r_{m+1}^p(x)\delta_{m+2}^p(x) - r_{m+2}^p(x)\delta_{m+1}^p(x) \rightarrow 0$$

as $m \rightarrow \infty$. Hence, by letting $m \rightarrow \infty$ in (6.29), we obtain

$$(6.31) \quad r_n^p(x)\delta_{n+1}^p(x) - r_{n+1}^p(x)\delta_n^p(x) = \sum_{j=n+1}^{\infty} \frac{S_j^p(x)r_j^p(x) + R_j^p(x)\delta_j^p(x)}{(j+\tau_0)^{p+3/2}}.$$

In a similar manner, we also have

$$(6.32) \quad s_n^p(x)\delta_{n+1}^p(x) - s_{n+1}^p(x)\delta_n^p(x) = \sum_{j=n+1}^{\infty} \frac{S_j^p(x)s_j^p(x) + S_j^p(x)\delta_j^p(x)}{(j+\tau_0)^{p+3/2}}$$

and

$$(6.33) \quad r_{n+1}^p(x)s_n^p(x) - s_{n+1}^p(x)r_n^p(x) = \frac{2}{\pi} + \sum_{j=n+1}^{\infty} \frac{S_j^p(x)r_j^p(x) - R_j^p(x)s_j^p(x)}{(j+\tau_0)^{p+3/2}}.$$

Then (6.26) follows from (6.31) and (6.32). The existence of a solution $\{\delta_n^p(x)\}_{n=1}^{\infty}$ to (6.26) is proved by using the method of successive approximation. Starting with $\delta_{n,0}^p(x) = 0$, we define $\delta_{n,k}^p(x)$ by

$$(6.34) \quad \begin{aligned} \delta_{n,k}^p(x) &= \sum_{j=n+1}^{\infty} \frac{[s_n^p(x)r_j^p(x) - r_n^p(x)s_j^p(x)]S_j^p(x)}{[s_{n+1}^p(x)r_n^p(x) - s_n^p(x)r_{n+1}^p(x)](j+\tau_0)^{p+\frac{3}{2}}} \\ &+ \sum_{j=n+1}^{\infty} \frac{[s_n^p(x)R_j^p(x) - r_n^p(x)S_j^p(x)]\delta_{j,k-1}^p(x)}{[s_{n+1}^p(x)r_n^p(x) - s_n^p(x)r_{n+1}^p(x)](j+\tau_0)^{p+\frac{3}{2}}} \end{aligned}$$

for $k \geq 1$ recursively. We shall show that for fixed p and sufficiently large but also fixed n , the sequence $\{\delta_{n,k}^p(x)\}_{k \geq 0}$ is convergent as $k \rightarrow \infty$. Since $\tilde{A}_s(\zeta)$

and $\tilde{B}_s(\zeta)$ are bounded for $t < t_2 - \sigma$, it follows from (6.6)–(6.9) that

$$(6.35) \quad |r_n^p(x)| \leq CN^{\frac{1}{2}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|J_\nu(N\zeta)| + |J_{\nu+1}(N\zeta)|]$$

and

$$(6.36) \quad |s_n^p(x)| \leq CN^{\frac{1}{2}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|W_\nu(N\zeta)| + |W_{\nu+1}(N\zeta)|]$$

for some positive constant C . Furthermore, by virtue of the behaviors of J_ν and W_ν , we have from (6.17), (6.19), (6.35) and (6.36) that

$$(6.37) \quad |R_n^p(x)s_n^p(x)| \leq M'N^{\frac{1}{2}} \quad \text{and} \quad |S_n^p(x)r_n^p(x)| \leq M'N^{\frac{1}{2}}$$

for some positive constant M' . It then follows from (6.33) that

$$\left| s_{n+1}^p(x)r_n^p(x) - r_{n+1}^p(x)s_n^p(x) + \frac{2}{\pi} \right| \leq \frac{2M'}{p} \cdot \frac{1}{N^p}.$$

Since the right-hand side tends to zero, this estimate gives

$$(6.38) \quad |s_{n+1}^p(x)r_n^p(x) - r_{n+1}^p(x)s_n^p(x)| > \frac{1}{\pi}$$

for large n . A combination of (6.34) and (6.35)–(6.38) yields

$$\begin{aligned} |\delta_{n,1}^p(x)| &\leq \pi \sum_{j=n+1}^{\infty} \frac{[|s_n^p(x)r_j^p(x)| + |r_n^p(x)s_j^p(x)|]|S_j^p(x)|}{(j + \tau_0)^{p+\frac{3}{2}}} \\ &\leq M'' \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} \sum_{j=n+1}^{\infty} \frac{N^{\frac{1}{2}}}{(j + \tau_0)^{p+1}} [|W_\nu(N\zeta)| + |W_{\nu+1}(N\zeta)|], \end{aligned}$$

where we have also made use of the monotonicity properties and the asymptotic behaviors of the modified Bessel functions I_ν and K_ν ; see (5.8), (5.11) and (5.12). Hence,

$$(6.39) \quad |\delta_{n,1}^p(x)| \leq \frac{M''}{p} \frac{1}{N^{p-\frac{1}{2}}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|W_\nu(N\zeta)| + |W_{\nu+1}(N\zeta)|].$$

Similarly, we can prove by induction that

$$(6.40) \quad \begin{aligned} &|\delta_{n,k}^p(x) - \delta_{n,k-1}^p(x)| \\ &\leq \left(\frac{M''}{p} \frac{1}{N^{p+1}} \right)^k N^{\frac{3}{2}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|W_\nu(N\zeta)| + |W_{\nu+1}(N\zeta)|], \end{aligned}$$

from which it follows that

$$(6.41) \quad \delta_{n,k}^p(x) = \sum_{m=1}^k [\delta_{n,m}^p(x) - \delta_{n,m-1}^p(x)]$$

converges as $k \rightarrow \infty$, for all $n > 2M'' - \tau_0$. Clearly, the limiting function $\delta_n^p(x)$ satisfies (6.26). Thus $Q_n(x) = s_n^p(x) + \delta_n^p(x)$ is a solution of (1.7) satisfying (6.20). Furthermore, we have from (6.39)–(6.41) that

$$|\delta_n^p(x)| \leq \frac{M'_p}{N^{p-\frac{1}{2}}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|W_\nu(N\zeta)| + |W_{\nu+1}(N\zeta)|].$$

Recall that $\tilde{A}_s(\zeta)$ and $\tilde{B}_s(\zeta)$ are bounded for $t < t_2 - \sigma$. By taking an extra term in the expansion in (6.2), we have

$$\begin{aligned} |\delta_n^p(x)| &\leq |\delta_n^{p+1}(x)| + \frac{M''_p}{N^{p+\frac{1}{2}}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} \\ (6.42) \quad &\quad \times [|W_\nu(N\zeta)| |\tilde{A}_{p+1}(\zeta)| + |W_{\nu+1}(N\zeta)| |\tilde{B}_{p+1}(\zeta)|] \\ &\leq \frac{M_p}{N^{p+\frac{1}{2}}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|W_\nu(N\zeta)| + |W_{\nu+1}(N\zeta)|] \end{aligned}$$

for some positive constants M''_p and M_p , and (2.21) follows. (Note that there is an extra factor of $N^{\frac{1}{2}}$ outside the square brackets in (2.19), so the error terms δ_n^p in (2.21) and (6.42) have a slightly different meaning.)

In a similar manner as we have done for $\delta_n^p(x)$, it can be proved that

$$(6.43) \quad |\varepsilon_n^p(x)| \leq \frac{M_p}{N^{p+\frac{1}{2}}} \left(\frac{4\zeta^2}{4 - (\alpha'_0 t + \beta'_0)^2} \right)^{\frac{1}{4}} [|J_\nu(N\zeta)| + |J_{\nu+1}(N\zeta)|],$$

and hence (2.20) follows. This completes the proof of Theorem 1.

7. AN EXAMPLE

As an illustration, we consider the orthogonal polynomials $p_n(x)$ associated with the Laguerre-type weights $w(x) = x^\alpha \exp(-q_m x^m)$, $x > 0$, $\alpha > -1$, $q_m > 0$. This weight is a Freud weight on the half line; the asymptotics of $p_n(x)$ have been investigated by Vanlessen [19] via a Riemann-Hilbert method. For the Freud weight on the whole real line, the asymptotics of the associated polynomials can be found in [9, 14, 26]. The notation $p_n(x) = \gamma_n x^n + \dots$ with leading coefficient $\gamma_n > 0$ denotes the n -th degree orthonormal polynomial with respect to $w(x)$. It is known that the sequence of $p_n(x)$ satisfies the three-term recurrence relation

$$(7.1) \quad x p_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x), \quad n = 1, 2, \dots,$$

with $p_{-1}(x) = 0$ and $p_0(x) = \gamma_0 > 0$. The recurrence coefficients a_n and b_{n-1} have the asymptotic expansions

$$(7.2) \quad b_{n-1} \sim n^{\frac{1}{m}} r_m \left\{ \frac{1}{4} + \frac{\alpha}{8m} \frac{1}{n} + \frac{c_2}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \right\}$$

and

$$(7.3) \quad a_n \sim n^{\frac{1}{m}} r_m \left\{ \frac{1}{2} + \frac{\alpha+1}{4m} \frac{1}{n} + \frac{d_2}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \right\},$$

where c_2 and d_2 are some constants and

$$(7.4) \quad r_m = \left(\frac{1}{2} m q_m \prod_{j=1}^m \frac{2j-1}{2j} \right)^{-1/m};$$

see [19]. We define a sequence $\{K_n\}$ by

$$K_n = n^{-\frac{1}{2m}} \prod_{l=0}^{\infty} \left[\left(\frac{n+2l}{n+2l+2} \right)^{\frac{1}{2m}} \cdot \frac{b_{n+2l+1}}{b_{n+2l}} \right].$$

Note that $K_{n+1}/K_{n-1} = b_{n-1}/b_n$ and $K_n \sim n^{-\frac{1}{2m}}$. We then put

$$(7.5) \quad A_n \equiv -\frac{1}{b_n} \frac{K_n}{K_{n+1}}, \quad B_n \equiv \frac{a_n}{b_n} \frac{K_n}{K_{n+1}}, \quad \mathcal{P}_n(x) \equiv (-1)^n [w(x)]^{\frac{1}{2}} p_n(x) \frac{1}{K_n}.$$

The three-term recurrence relation in (7.1) now becomes

$$(7.6) \quad \mathcal{P}_{n+1}(x) + \mathcal{P}_{n-1}(x) = (A_n x + B_n) \mathcal{P}_n(x).$$

A slight computation gives the following asymptotic expansions

$$(7.7) \quad \frac{K_n}{K_{n+1}} \sim 1 + \frac{1}{2m} \frac{1}{n} - \frac{4m+2\alpha m-1}{8m^2} \frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right),$$

$$(7.8) \quad A_n \sim n^{-\frac{1}{m}} r_m^{-1} \left\{ -4 + \frac{2+2\alpha}{m} \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right\}$$

and

$$(7.9) \quad B_n \sim 2 + \frac{(4m^2 - 4m + 1)\alpha^2 - m^2}{4m^2} \frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right).$$

In terms of our notations, $\theta = \frac{1}{m}$ and $\tau_0 = \frac{\alpha+1}{2}$. Set $N = n + \tau_0$. We then have

$$(7.10) \quad A_n \sim N^{-\frac{1}{m}} \sum_{s=0}^{\infty} \frac{\alpha'_s}{N^s}, \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{N^s},$$

where

$$\alpha'_0 = -4r_m^{-1}, \quad \alpha'_1 = 0, \quad \beta'_0 = 2, \quad \beta'_1 = 0, \quad \beta'_2 = \frac{(2m-1)^2 \alpha^2 - m^2}{4m^2}.$$

Let $x = N^\theta t$. To simplify matters, we further set $t = r_m z$ and $\omega_n = N^\theta r_m$, so that $x = \omega_n z$. Since $\theta = \frac{1}{m} \neq 0, 2$ and $\beta'_0 = 2$, the two transition points are $t_1 = 0$ and $t_2 = r_m$; cf. (1.11). We can now apply Theorem 1. In terms of the

new scaled variable z , the two transition points are $z_1 = 0$ and $z_2 = 1$. From (2.15), it can be shown that $\nu = \alpha$, and by (4.9)

$$(7.11) \quad \zeta(z) = \arccos(1 - 2z) + \frac{2\sqrt{z(1-z)}}{2m-1} {}_2F_1\left(1, 1-m; \frac{3}{2}-m; z\right), \quad z < 1 - \sigma.$$

Remark 2. For any given integer $m > 0$, the hypergeometric function in (7.11) is a polynomial of degree $(m-1)$. For example, when $m = 1$, it is 1; when $m = 2$, it is $2z + 1$; and when $m = 3$, it is $\frac{8}{3}z^2 + \frac{4}{3}z + 1$.

From Theorem 1, it follows that we have two linearly independent solutions $P_n(x)$ and $Q_n(x)$ given in (2.18) and (2.19). Hence,

$$(7.12) \quad \mathcal{P}_n(\omega_n z) = C_1(x)P_n(x) + C_2(x)Q_n(x),$$

where $C_1(x)$ and $C_2(x)$ are functions depending only on x . To determine $C_1(x)$ and $C_2(x)$, we note that in terms of our notations, equation (2.12) in [19] reads

$$(7.13) \quad [w(\beta_n z)]^{\frac{1}{2}} p_n(\beta_n z) \sim \sqrt{\frac{2}{\pi\beta_n} \frac{[(2z-1) + 2\sqrt{z(z-1)}]^{\frac{\alpha+1}{2}}}{2z^{\frac{1}{4}}(z-1)^{\frac{1}{4}}}} \exp\{-in\zeta(z)\},$$

where $\beta_n = n^{\frac{1}{m}} r_m$. This asymptotic formula holds uniformly for z in any complex subset $K \subset \mathbb{C} \setminus [0, 1]$. Note that our uniform asymptotic expansion of $\mathcal{P}_n(\omega_n z)$ (7.12) is also valid in a neighbourhood of any subinterval of $(-\infty, 1 - \sigma]$ in the complex z -plane. Since $\omega_n = (1 + \tau_0/n)^{\frac{1}{m}} \beta_n$, matching these two asymptotic formulas in an overlapping region, and using the behaviors of $J_\nu(x)$ and $Y_\nu(x)$ for large x , we obtain $C_1(x) = 1$ and $C_2(x) = 0$. That is

$$(7.14) \quad \mathcal{P}_n(\omega_n z) = N^{\frac{1}{2}} \left(\frac{\zeta^2(z)}{z(1-z)} \right)^{\frac{1}{4}} \left[J_\alpha(N\zeta) \sum_{s=0}^p \frac{\tilde{A}_s(\zeta)}{N^s} + J_{\alpha+1}(N\zeta) \sum_{s=0}^p \frac{\tilde{B}_s(\zeta)}{N^s} + \varepsilon_n^p \right],$$

where $\tilde{A}_0(\zeta) = 1$, $\tilde{B}_0(\zeta) = 0$, and

$$(7.15) \quad |\varepsilon_n^p| \leq \frac{M_p}{N^{p+1}} [|J_\alpha(N\zeta)| + |J_{\alpha+1}(N\zeta)|]$$

for all $z \leq 1 - \sigma$. We can also use the result in [21] to derive a uniform asymptotic expansion for $\mathcal{P}_n(x)$ near the turning point $z_2 = 1$, and we have

$$(7.16) \quad (-1)^n \mathcal{P}_n(\omega_n z) = N^{\frac{1}{6}} \left(\frac{\eta(z)}{z(z-1)} \right)^{\frac{1}{4}} \left[\text{Ai}(N^{\frac{2}{3}}\eta) \sum_{s=0}^p \frac{\overline{A}_s(\eta)}{N^s} + \text{Ai}'(N^{\frac{2}{3}}\eta) \sum_{s=0}^p \frac{\overline{B}_s(\eta)}{N^{s+1/3}} + \overline{\varepsilon}_n^p \right]$$

for $z \geq \sigma > 0$, where $\overline{A}_0(\eta) = 1$, $\overline{B}_0(\eta) = 0$,

$$(7.17) \quad \frac{2}{3}[-\eta(z)]^{\frac{3}{2}} = \arccos(2z - 1) - \frac{2\sqrt{z(1-z)}}{2m-1} \cdot {}_2F_1\left(1, 1-m; \frac{3}{2}-m; z\right)$$

for $\sigma \leq z \leq 1$, and $\eta(z)$ analytically continued to $\mathbb{C} \setminus \{(-\infty, 0] \cup [1, -i\infty)\}$ such that $\eta(z) > 0$ for $z > 1$. The error estimation is given by

$$(7.18) \quad |\overline{\varepsilon}_n^p| \leq \frac{M_p}{N^{p+1}} \widetilde{\text{Ai}}(N^{\frac{2}{3}}\eta)$$

with the modulus function $\widetilde{\text{Ai}}(N^{\frac{2}{3}}\eta)$ as defined in [21, eq. (7.10)]. Note that the two results in (7.14) and (7.16) together cover the whole real line.

Remark 3. *In the special case of $m = 1$ and $q_m = 1$, we get two uniform asymptotic expansions for Laguerre polynomials, which agree with the results obtained by steepest decent method for integrals [11] or WKB approximations for differential equations [15, Chaps. 11 and 12].*

For orthogonal polynomials associated with the weight $x^\alpha \exp(-Q(x))$, $x > 0$, $\alpha > -1$ and $Q(x)$ is a polynomial of m -th degree with positive leading coefficient, the asymptotic expansions of the recurrence coefficients will be in powers of $1/n^{\frac{1}{m}}$, instead of in powers series of $1/n$; cf. (1.8). In such a case, one can modify the method provided in this paper to get a pair of linearly independent solutions to the three-term recurrence relation given in Theorem 1, which are also in terms of Bessel functions near the transition point $t_1 = 0$. However, the asymptotic expansions for these solutions are on longer in powers of $1/N$, but in powers of $1/N^{\frac{1}{m}}$. This result will lead to Bessel-type asymptotic expansions given by Vanlessen in [19].

As another example, confluent hypergeometric functions $M(a, b; x)$ and $\Gamma(a - b + 1) \cdot U(a, b; x)$ satisfy the three-term recurrence relation in a :

$$(7.19) \quad ay(a+1, b; x) + (a-b)y(a-1, b; x) = (2a-b+x)y(a, b; x).$$

A straightforward calculation shows that equation (7.19) corresponds to our case for $\theta = 1$, $\alpha_0 = 1$ and $\beta_0 = 2$. A direct application of Theorem 1 yields Bessel-type expansions as $a \rightarrow +\infty$. Replacing a by $-a$ in (7.19), one can obtain again, by the main theorem, Bessel-type expansions for $M(a, b; x)$ or $U(a, b; x)$ as $a \rightarrow -\infty$; see [16, §13.8(iii)].

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